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Queueing system $M/M/n/(m,V)$ with non-identical servers

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Abstract

We investigate multi-server queueing systems with Poisson arrivals, non-identical servers and customers of random volume, under assumption that customer's service time having an exponential distribution doesn't depend on his volume, but service time parameters can be different for different servers. We also assume that the total volume of customers present in the system at arbitrary time instant is bounded by some constant value $V > 0$.

For such systems the stationary customers number distribution and loss probability are determined.

1. Analysis of $M/M/n/(m,V)$ queueing system with identical servers

Consider the system $M/M/n/(m,V)$ with identical servers [6].

Denote by $\eta(t)$ the number of customers present in the system at time instant t. Let $\sigma_1(t), \sigma_2(t), \ldots, \sigma_{\eta(t)}(t)$ be the volumes of customers numbered by $1, 2, \ldots, \eta(t)$ according to their coming to the system; a be the parameter of Poisson arrival flow and μ be the parameter of service time distribution. Let $L(x) = P\{\zeta < x\}$ be the distribution function of customers volume ζ that is a non-negative random variable.

Then we can describe the system under consideration by the following markovian process:

$$\left(\eta(t), \sigma_1(t), \sigma_2(t), \ldots, \sigma_{\eta(t)}(t)\right). \tag{1}$$

Process (1) can be characterized by the following functions:

$$P_k(t) = P\{\eta(t) = k\}, k = \overline{0, n+m}, \tag{2}$$

$$G_k(t, x) = P\{\eta(t) = k, \sigma(t) < x\}, k = \overline{1, n+m}, \tag{3}$$

where $\sigma(t) = \sum_{i=1}^{\eta(t)} \sigma_i(t)$ is the total volume of customers present in the system at time instant t.

It is clear that for $k = \overline{1, n+m}$ we have the relation

$$P_k(t) = G_k(t, V). \tag{4}$$

For the functions (2), (3) we can write down the following equations:

$$P_0'(t) = -aP_0(t)L(V) + \mu P_1(t); \tag{5}$$

$$P_1'(t) = aP_0(t)L(V) - a\int_0^V G_1(t, V-y)dL(y) - \mu P_1(t) + 2\mu P_2(t); \tag{6}$$

$$P_k'(t) = a\int_0^V G_{k-1}(t, V-y)dL(y) - a\int_0^V G_k(t, V-y)dL(y) -$$
$$-k\mu P_k(t) + (k+1)\mu P_{k+1}(t),\ k = \overline{2, n-1}; \tag{7}$$

$$P_k'(t) = a\int_0^V G_{k-1}(t, V-y)dL(y) - a\int_0^V G_k(t, V-y)dL(y) -$$
$$-n\mu P_k(t) + n\mu P_{k+1}(t),\ k = \overline{n, n+m-1}; \tag{8}$$

$$P_{n+m}'(t) = a\int_0^V G_{n+m-1}(t, V-y)dL(y) - n\mu P_{n+m}(t). \tag{9}$$

In stationary mode that exists if $\rho = a/(n\mu) < \infty$, we can introduce the following stationary analogies of the functions (2), (3):

$$p_k = P\{\eta = k\}, k = \overline{0, n+m}, \tag{10}$$

$$g_k(x) = P\{\eta = k, \sigma < x\}, k = \overline{1, n+m}, \tag{11}$$

where $\eta(t) \Rightarrow \eta$ and $\sigma(t) \Rightarrow \sigma$ in the sense of a weak convergences.

Then the steady state equations for the functions (10), (11) follow from the equations (5)–(9) and take the form

$$0 = -ap_0L(V) + \mu p_1; \tag{12}$$

$$0 = ap_0L(V) - a\int_0^V g_1(V-y)dL(y) - \mu p_1 + 2\mu p_2; \tag{13}$$

$$0 = a\int_0^V g_{k-1}(V-y)dL(y) - a\int_0^V g_k(V-y)dL(y) - \\ -k\mu p_k + (k+1)\mu p_{k+1},\ k = \overline{2, n-1}; \tag{14}$$

$$0 = a\int_0^V g_{k-1}(V-y)dL(y) - a\int_0^V g_k(V-y)dL(y) - \\ -n\mu p_k + n\mu p_{k+1},\ k = \overline{n, n+m-1}; \tag{15}$$

$$0 = a\int_0^V g_{n+m-1}(V-y)dL(y) - n\mu p_{n+m}. \tag{16}$$

Let us introduce the notation $L_k(y)$ for kth order Stieltjes convolution of the distribution function $L(y)$, which is defined recurrently as follows:

$$L_0(y) \equiv 1,\ L_k(y) = \int_0^y L_{k-1}(y-u)dL(u).$$

In addition, we introduce the notation

$$N(k) = \begin{cases} \frac{(n\rho)^k}{k!}, \text{ if } k = \overline{1, n}; \\ \frac{n^n\rho^k}{n!}, \text{ if } k = \overline{n+1, n+m}. \end{cases}$$

By direct substitution, we can check that the solution of (12)–(16) has the form

$$g_k(x) = p_0N(k)L_k(x), k = \overline{1, n+m}.$$

By the limiting transition in (4), we can obtain formulas for p_k:

$$p_k = g_k(V) = p_0N(k)L_k(V), k = \overline{1, n+m}. \tag{17}$$

From the normalization condition $\sum_{k=0}^{n+m} p_k = 1$ we also obtain

$$p_0 = \left[1 + \sum_{k=1}^{n+m} N(k)L_k(V)\right]^{-1}. \tag{18}$$

The loss probability can be obtained from the following equilibrium condition:

$$a(1-p_u) = \mu\sum_{k=1}^{n-1} kp_k + n\mu(1 - \sum_{k=0}^{n-1} p_k),$$

whereas we have

$$p_u = 1 - (n\rho)^{-1}\sum_{k=1}^{n-1} kp_k - \rho^{-1}(1 - \sum_{k=0}^{n-1} p_k), \qquad (19)$$

where probabilities p_k are determined by (17). The results for analyzed system were presented, for example, in [6].

2. $M/M/n/(m, V)$ queueing system with non-identical servers and the random choice of a server

In this section we present some generalization of the system discussed in section 1. The purpose of our investigations is to obtain formulas for probabilities p_k and loss probabilities in the steady state and to analyze some special cases. We use some classical results for $M/M/n/m$ queuing systems with non–identical servers [1–5, 8] and some basic properties of queueing systems with non-homogeneous customers and customer's service time independent on its volume [6, 7].

If the parameters of service time distribution are not identical for every server, then the behaviour of the system is described by the following markovian process:

$$\left(\eta(t), i_1(t), i_2(t), \ldots, i_l(t), \sigma_1(t), \sigma_2(t), \ldots, \sigma_{\eta(t)}(t)\right), \qquad (20)$$

where $l = \min(\eta(t), n)$ and $i_1(t), i_2(t), \ldots, i_l(t)$ is the sequence of the numbers of busy servers ordered increasingly. If $\eta(t) = 0$, the process (20) reduces to $\eta(t)$.

Process (20) is characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \qquad (21)$$

$$G_{kf_1f_2\ldots f_l}(t, x) = P\{\eta(t) = k,\ i_1(t) = f_1, i_2(t) = f_2, \ldots, i_l(t) = f_l, \sigma(t) < x\},$$
$$k = \overline{1, n+m}. \qquad (22)$$

It is obvious that for $k \geq n$ function (22) can be rewritten as

$$G_k(t, x) = P\{\eta(t) = k, \sigma(t) < x\}.$$

If $k < n$, we have

$$G_k(t, x) = P\{\eta(t) = k, \sigma(t) < x\} = \sum_{\{F_k^n\}} G_{kf_1f_2\ldots f_k}(t, x), \qquad (23)$$

where $\{F_k^n\}$ is the set of all k-element combinations of the set $\{f_1, f_2, \ldots, f_n\}$.

Assume additionally that we have only two non-identical servers. Denote as μ_1, μ_2 time service parameters for first and the second server consequently. Let us introduce the notation $\rho = \frac{a}{\mu_1+\mu_2}$.

In this case the process (20) take the form

$$\left(\eta(t), i_1(t), \ldots, i_l(t), \sigma_1(t), \sigma_2(t), \ldots, \sigma_{\eta(t)}(t)\right), \tag{24}$$

where $l = \min(\eta(t), 2)$.

Process (24) can be characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \tag{25}$$

$$P_{11}(t) = P\{\eta(t) = 1, i_1(t) = 1\}; \tag{26}$$

$$P_{12}(t) = P\{\eta(t) = 1, i_1(t) = 2\}; \tag{27}$$

$$P_k(t) = P\{\eta(t) = k\}, k = \overline{2, m+2}; \tag{28}$$

$$G_{11}(t,x) = P\{\eta(t) = 1, i_1(t) = 1, \sigma(t) < x\}; \tag{29}$$

$$G_{12}(t,x) = P\{\eta(t) = 1, i_1(t) = 2, \sigma(t) < x\}; \tag{30}$$

$$G_k(t,x) = P\{\eta(t) = k, \sigma(t) < x\},\ k = \overline{2, m+2}. \tag{31}$$

If we analyze the behaviour of the system, we can write down the following equations:

$$P_0'(t) = -aP_0(t)L(V) + \mu_1 P_{11}(t) + \mu_2 P_{12}(t); \tag{32}$$

$$P_{11}'(t) = -a\int_0^V G_{11}(t, V-x)dL(x) - \mu_1 P_{11}(t) + \frac{a}{2}P_0(t)L(V) + \mu_2 P_2(t); \tag{33}$$

$$P_{12}'(t) = -a\int_0^V G_{12}(t, V-x)dL(x) - \mu_2 P_{12}(t) + \frac{a}{2}P_0(t)L(V) + \mu_1 P_2(t); \tag{34}$$

$$P_2'(t) = -a\int_0^V G_2(t, V-x)dL(x) - (\mu_1+\mu_2)P_2(t) +$$

$$+a\left(\int_0^V G_{11}(t, V-x)dL(x) + \int_0^V G_{12}(t, V-x)dL(x)\right) + (\mu_1+\mu_2)P_3(t); \tag{35}$$

$$P_k'(t) = -a\int_0^V G_k(t, V-x)dL(x) - (\mu_1+\mu_2)P_k(t) +$$

$$+a\int_0^V G_{k-1}(t, V-x)dL(x) + (\mu_1+\mu_2)P_{k+1}(t),\ k = \overline{3, m+1}; \tag{36}$$

$$P'_{m+2}(t) = -(\mu_1+\mu_2)P_{m+2}(t) + a\int_0^V G_{m+1}(t, V-x)dL(x); \tag{37}$$

$$P_0(t) + P_{11}(t) + P_{12}(t) + \sum_{k=2}^{m+2} P_k(t) = 1. \tag{38}$$

In the steady state (if only $\rho < \infty$), we obtain the following equations for functions p_0, p_{11}, p_{12}, p_k, $g_{11}(x)$, $g_{12}(x)$, $g_k(x)$, that are the limits of functions (25)–(31) (if $t \to \infty$) in the sens of weak convergence:

$$0 = -ap_0L(V) + \mu_1 p_{11} + \mu_2 p_{12}; \tag{39}$$

$$0 = -a\int_0^V g_{11}(V-x)dL(x) - \mu_1 p_{11} + \frac{a}{2}p_0L(V) + \mu_2 p_2; \tag{40}$$

$$0 = -a\int_0^V g_{12}(V-x)dL(x) - \mu_2 p_{12} + \frac{a}{2}p_0L(V) + \mu_1 p_2; \tag{41}$$

$$0 = -a\int_0^V g_2(V-x)dL(x) - (\mu_1+\mu_2)p_2 +$$

$$+a\left(\int_0^V g_{11}(V-x)dL(x) + \int_0^V g_{12}(V-x)dL(x)\right) + (\mu_1+\mu_2)p_3; \tag{42}$$

$$0 = -a\int_0^V g_k(V-x)dL(x) - (\mu_1+\mu_2)p_k +$$

$$+a\int_0^V g_{k-1}(V-x)dL(x) + (\mu_1+\mu_2)p_{k+1},\ k = \overline{3, m+1}; \tag{43}$$

$$0 = -(\mu_1+\mu_2)p_{m+2} + a\int_0^V g_{m+1}(V-x)dL(x); \tag{44}$$

$$p_0 + p_{11} + p_{12} + \sum_{k=2}^{m+2} p_k = 1. \tag{45}$$

By the direct substitution, we can check that the solution of (39)–(45) has the form

$$g_{11}(x) = \frac{a}{2\mu_1}p_0L(x),\ g_{12}(x) = \frac{a}{2\mu_2}p_0L(x); \tag{46}$$

$$g_k(x) = \frac{ap_0}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\rho^{k-1}L_k(x),\ k = \overline{2, m+2}; \tag{47}$$

$$p_{11} = \frac{a}{2\mu_1}p_0L(V),\ p_{12} = \frac{a}{2\mu_2}p_0L(V); \tag{48}$$

$$p_1 = p_{11} + p_{12} = \frac{ap_0}{2}(\frac{1}{\mu_1} + \frac{1}{\mu_2})L(V); \tag{49}$$

$$p_k = \frac{ap_0}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\rho^{k-1}L_k(V),\ k = \overline{2, m+2}. \tag{50}$$

The formulas (49)–(50) can be rewritten as it follows:

$$p_k = \frac{ap_0}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\rho^{k-1}L_k(V),\ k = \overline{1, m+2}, \tag{51}$$

where p_0 can be obtained from the normalization condition (45) and has the form

$$p_0 = \left[1 + \frac{a}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\sum_{k=1}^{m+2}\rho^{k-1}L_k(V)\right]^{-1}. \tag{52}$$

The above analysis can be generalized for the arbitrary number of non-identical servers

For example, in the case of $n = 3$ we obtain

$$p_{11} = \frac{ap_0}{3\mu_1}L(V),\ p_{12} = \frac{ap_0}{3\mu_2}L(V),\ p_{13} = \frac{ap_0}{3\mu_3}L(V); \tag{53}$$

$$p_1 = p_{11} + p_{12} + p_{13} = \frac{ap_0}{3}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right)L(V); \tag{54}$$

$$p_{212} = \frac{a^2p_0}{6\mu_1\mu_2}L_2(V),\ p_{213} = \frac{a^2p_0}{6\mu_1\mu_3}L_2(V),\ p_{223} = \frac{a^2p_0}{6\mu_2\mu_3}L_2(V); \tag{55}$$

$$p_2 = p_{212} + p_{213} + p_{223} = \frac{a^2p_0}{6}\left(\frac{1}{\mu_1\mu_2} + \frac{1}{\mu_1\mu_3} + \frac{1}{\mu_2\mu_3}\right)L_2(V); \tag{56}$$

$$p_k = \frac{a^2p_0}{6}\left(\frac{1}{\mu_1\mu_2} + \frac{1}{\mu_1\mu_3} + \frac{1}{\mu_2\mu_3}\right)\rho^{k-2}L_k(V),\ k = \overline{3, m+3}, \tag{57}$$

where $\rho = \frac{a}{\mu_1+\mu_2+\mu_3}$. Formulas (56)–(57) can be rewritten as

$$p_k = \frac{a^2p_0}{6}\left(\frac{1}{\mu_1\mu_2} + \frac{1}{\mu_1\mu_3} + \frac{1}{\mu_2\mu_3}\right)\rho^{k-2}L_k(V),\ k = \overline{2, m+3}. \tag{58}$$

where we obtain the following value for p_0:

$$p_0 = \left[1 + \frac{a}{3}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right)L(V) + \right.$$

$$\left. + \frac{a^2}{6}\left(\frac{1}{\mu_1\mu_2} + \frac{1}{\mu_1\mu_3} + \frac{1}{\mu_2\mu_3}\right)\sum_{k=2}^{m+3}\rho^{k-2}L_k(V)\right]^{-1}. \tag{59}$$

The solutions of analyzed systems of equations in the steady state are obtained using computer algebra systems (ex. *Mathematica* environment).

In general case we obtain the following formulas:

$$p_{kf_1f_2\dots f_k} = \frac{a^k(n-k)!p_0}{n!\prod_{i=1}^{k}\mu_{f_i}} L_k(V), k = \overline{1, n-1}; \tag{60}$$

$$p_k = \begin{cases} \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}L_k(V), \ k = \overline{1, n-2}, \\ \frac{a^{n-1}p_0}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}\rho^{k-n+1}L_k(V), \ k = \overline{n-1, n+m}, \end{cases} \tag{61}$$

where F_k^n denotes k-element subset of n-element set and

$$p_0 = \left[1 + \frac{1}{n!}\sum_{k=1}^{n-2}a^k(n-k)!\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}L_k(V) + \right.$$

$$\left. + \frac{a^{n-1}}{n!}\sum_{k=n-1}^{n+m}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}\rho^{k-n+1}L_k(V)\right]^{-1}. \tag{62}$$

3. Loss probability

Assume that we have a system with two non-identical servers. Denote as μ_1, μ_2 service time parameters for the first and second server consequently. Denote as p_u the loss probability for the system under consideration. To obtain the value of p_u we can write down the following equilibrium condition:

$$a(1-p_u) = \mu_1 p_{11} + \mu_2 p_{12} + (\mu_1+\mu_2)\left(1 - \sum_{k=0}^{1} p_k\right). \tag{63}$$

In the case of three servers the equilibrium condition has the form

$$a(1-p_u) = \mu_1 p_{11} + \mu_2 p_{12} + \mu_3 p_{13} + (\mu_1+\mu_2)p_{212} +$$

$$+ (\mu_1+\mu_3)p_{213} + (\mu_2+\mu_3)p_{223} + (\mu_1+\mu_2+\mu_3)\left(1 - \sum_{k=0}^{2} p_k\right). \tag{64}$$

In general we have the following formula

$$a(1-p_u) = \sum_{k=1}^{n-1}\sum_{\{F_k^n\}} p_{kf_1f_2\dots f_k}\sum_{i=1}^{k}\mu_{f_i} + \sum_{k=1}^{n}\mu_k\left(1 - \sum_{k=0}^{n-1} p_k\right). \tag{65}$$

The solution of (65) leads to the following result:

$$p_u = 1 - \frac{1}{a}\left[\sum_{k=1}^{n-1}\sum_{\{F_k^n\}} p_{kf_1f_2\ldots f_k}\sum_{i=1}^{k}\mu_{f_i} + \sum_{k=1}^{n}\mu_k\left(1-\sum_{k=0}^{n-1}p_k\right)\right], \qquad (66)$$

where $p_{kf_1f_2\ldots f_k}$ and p_k are determined by relations (60) and (61).

4. Analysis of some special cases

1. $M/M/n/(0,V)$ **queueing system with non-identical servers.**

Consider now a queueing system with no waiting places in the queue ($m = 0$). In this case formula (61) has the form

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}L_k(V), \;\; k=\overline{1,n-1};$$

$$p_n = \frac{a^{n-1}p_0\rho}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}L_n(V),$$

where

$$p_0 = \left[1+\frac{1}{n!}\sum_{k=1}^{n-1}a^k(n-k)!\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}L_k(V)+\right.$$

$$\left.+\frac{a^{n-1}\rho}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}L_n(V)\right]^{-1}.$$

Loss probability on the base of the relation (66) has the form

$$p_u = 1-\frac{1}{a}\left(\sum_{k=1}^{n-1}\sum_{\{F_k^n\}}p_{kf_1f_2\ldots f_k}\sum_{i=1}^{k}\mu_{f_i}+p_n\sum_{k=1}^{n}\mu_k\right).$$

Assume additionally that customer's volume has an exponential distribution with the parameter f i.e. $L(x) = 1-e^{-fx}$. In this case we have $L_k(x) = 1-e^{-fx}\sum_{i=0}^{k-1}\frac{(fx)^i}{i!}$ i.e. the function $L_k(x)$ has the $k-$Erlang distribution with the parameter f. So we finally obtain the formulas

$$p_{kf_1f_2\dots f_k} = \frac{a^k(n-k)!p_0}{n!\prod_{i=1}^k \mu_{f_i}}\left(1 - e^{-fV}\sum_{i=0}^{k-1}\frac{(fV)^i}{i!}\right),\ k = \overline{1, n-1};$$

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}\left(1 - e^{-fV}\sum_{i=0}^{k-1}\frac{(fV)^i}{i!}\right),\ k = \overline{1, n-1};$$

$$p_n = \frac{a^{n-1}p_0\rho}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}\left(1 - e^{-fV}\sum_{i=0}^{n-1}\frac{(fV)^i}{i!}\right). \qquad (67)$$

Assume now that customers's volume has geometric distribution with the parameter f i.e. $L(x) = \sum_{k<x}(1-f)f^k$. Then we have

$$L_k(x) = \sum_{i<x}\binom{k+i-1}{i}f^i(1-f)^k.$$

So we obtain the following results:

$$p_{kf_1f_2\dots f_k} = \frac{a^k(n-k)!p_0}{n!\prod_{i=1}^k \mu_{f_i}}\sum_{i<V}\binom{k+i-1}{i}f^i(1-f)^k,\ k = \overline{1, n-1};$$

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}\sum_{i<V}\binom{k+i-1}{i}f^i(1-f)^k,\ k = \overline{1, n-1};$$

$$p_n = \frac{a^{n-1}p_0\rho}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}\sum_{i<V}\binom{n+i-1}{i}f^i(1-f)^n. \qquad (68)$$

Finally we assume that customer's volume is constant i.e. $\zeta = f_0$. Then we have $L(x) = H(x - f_0)$, where $H(x)$ is the Heaviside unitstep function and $L_k(x) = H(\frac{x}{k} - f_0)$.
So we obtain the following results:

$$p_{kf_1f_2\dots f_k} = \frac{a^k(n-k)!p_0}{n!\prod_{i=1}^k \mu_{f_i}}H\left(\frac{V}{k} - f_0\right), k = \overline{1, n-1};$$

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}H\left(\frac{V}{k} - f_0\right),\ k = \overline{1, n-1};$$

$$p_n = \frac{a^{n-1}p_0\rho}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}H\left(\frac{V}{n}-f_0\right). \tag{69}$$

2. $M/M/n/(\infty,V)$ **queueing system with non-identical servers.**

Consider now queueing system with infinite number of waiting places ($m=\infty$). In this case formula (61) has the form

$$p_k = \begin{cases} \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}L_k(V), \ k=\overline{1,n-2}, \\ \frac{a^{n-1}p_0}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}\rho^{k-n+1}L_k(V), \ k\ge n-1, \end{cases}$$

where

$$p_0 = \left[1+\frac{1}{n!}\sum_{k=1}^{n-2}a^k(n-k)!\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}L_k(V)+\right.$$

$$\left.+\sum_{k=n-1}^{\infty}\frac{a^{n-1}}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}\rho^{k-n+1}L_k(V)\right]^{-1}.$$

In addition, loss probability is determined by (66).

Assume additionally that customer's volume has exponential distribution with the parameter f. Then we have the following formulas:

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}\left(1-e^{-fV}\sum_{i=0}^{k-1}\frac{(fV)^i}{i!}\right), \ k=\overline{1,n-2};$$

$$p_k = \frac{a^{n-1}p_0\rho^{k-n+1}}{n!}\sum_{\{F_{n-1}^n\}}\frac{1}{\prod_{x_i\in F_{n-1}^n}\mu_{x_i}}\left(1-e^{-fV}\sum_{i=0}^{k-1}\frac{(fV)^i}{i!}\right), \ k\ge n-1. \tag{70}$$

Assume now that customers's volume has geometric distribution with the parameter f. Then we have

$$p_k = \frac{a^k(n-k)!p_0}{n!}\sum_{\{F_k^n\}}\frac{1}{\prod_{x_i\in F_k^n}\mu_{x_i}}\sum_{i<V}\binom{k+i-1}{i}f^i(1-f)^k, \ k=\overline{1,n-2};$$

$$p_k = \frac{a^{n-1} p_0 \rho^{k-n+1}}{n!} \sum_{\{F_{n-1}^n\}} \frac{1}{\prod_{x_i \in F_{n-1}^n} \mu_{x_i}} \sum_{i<V} \binom{k+i-1}{i} f^i (1-f)^k, \ k \geq n-1. \tag{71}$$

If the customer's volume is constant ($\zeta = f_0$), then we obtain the following results:

$$p_k = \frac{a^k (n-k)! p_0}{n!} \sum_{\{F_k^n\}} \frac{1}{\prod_{x_i \in F_k^n} \mu_{x_i}} H\left(\frac{V}{k} - f_0\right), \ k = \overline{1, n-2};$$

$$p_k = \frac{a^{n-1} p_0 \rho^{k-n+1}}{n!} \sum_{\{F_{n-1}^n\}} \frac{1}{\prod_{x_i \in F_{n-1}^n} \mu_{x_i}} H\left(\frac{V}{k} - f_0\right), \ k \geq n-1 \tag{72}$$

References

[1] D. V. Efrosinin, V. V. Rykov, *Numerical Study of the Optimal Control of a System with Heterogeneous Servers,* Automation and Remote Control, vol. 64, no. 2, pp. 302-309, 2003.

[2] D. V. Efrosinin, V. V. Rykov, *On Performance Characteristics for Queueing Systems with Heterogeneous Servers,* Automation and Remote Control, vol. 69, no. 1, pp. 61-75, 2008.

[3] H. Gumbel, *Waiting lines with heterogeneous servers,* Operations Research, vol. 8, no. 4, pp. 504-511, 1960.

[4] V. P. Singh, *Markovian queues with three heterogeneous servers,* AIIE Transactions, vol. 3, no. 1, pp. 45-48, 1971.

[5] V. P. Singh, *Two-server Markovian queues with balking: Heterogeneous vs. homogeneous servers,* Operation Research, vol. 18, no. 1, pp. 145-159, 1970.

[6] O. Tikhonenko, *Metody probabilistyczne analizy systemów informacyjnych,* Akademicka Oficyna Wydawnicza EXIT, Warszawa, 2006.

[7] O. Tikhonenko, M. Kawecka, *Systemy obsługi zgłoszeń o losowej objętości z ograniczoną objętością pamięci,* Studia Informatica, vol. 33, no. 3A, pp. 19-28, 2012.

[8] M. Ziólkowski, *$M/M/n/m$ queueing system with non-identical servers,* Scientific issues of Jan Długosz University in Częstochowa, Mathematics vol. 16, pp. 163-168, 2011.