

DIVISIBILITY AND ITS APPLICATION IN TEACHING MATHEMATICS IN THE THIRD EDUCATIONAL LEVEL

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ABSTRACT

Elements of theory of divisibility are present in many various interesting mathematical tasks, especially in tasks that are addressed to talented pupils taking part in mathematical competitions. Good understanding of it lets solve very interesting and difficult (at first glance) issues. On the other hand, there are a lot of problems with understanding such terms as: multiple, divisor, divisibility, prime number, LCM, GCD etc. The purpose of the article is presenting the base terms of theory in the language understood for pupil of the gymnasium (3-rd educational level, 13-16 years old). In addition, we present some algorithms that are used to solve problems from the theory of divisibility and we discuss the influence the choice of the algorithm on its effectiveness (so we analyze its complexity). Presented algorithms let us create computer programs that solve the problems mechanically. We also enlarge a bit some topics for those ones which can be taught in the class of pupils which are interested in Mathematics.

1. DIVISIBILITY AND MULTIPLES

Everybody meets divisibility in the second educational level. Children are taught that some positive integers can be divided by some positive integers, but some of them not. We can say that a positive integer n can be divided by a positive integer m with a remainder. It means that for n and m there exist integers k and r such that

$$\frac{n}{m} = k + \frac{r}{m}, \quad \text{where } r = 0, 1, \dots, m - 1.$$

In other words:

$$n = k \cdot m + r, \quad \text{where } r = 0, 1, \dots, m - 1.$$

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If we define divisibility in the following way: a number m is a divisor of a number n if the remainder from the division n by m equals 0, then the word *divisor* has some connection with the word *divide*. In fact, we apply here the following theorem:

Theorem 1. *For any nonnegative integer n and positive integer m there exist nonnegative integers k and r such that*

$$n = k \cdot m + r, \quad \text{where } r = 0, 1, \dots, m - 1.$$

If the remainder equals 0, then this theorem says that

$$n = k \cdot m.$$

And this equality plays the main role in the definition of divisibility on the higher levels. The students are forgetting the prime idea of divisibility, so it can be one of the mistakes in understanding this notion defined in the following way:

Definition 1. *An integer n is divisible by positive integer m if there exists an integer k such that $n = k \cdot m$. In that case m is called a divisor of n .*

This definition can be generalized for all integers, for elements from some rings and so on. It is pretty useful, however can cause some problems for our students since the word “divisible” is similar (especially in polish language) to the word “division”. But there is no division, multiplication only, in this definition.

One could state from the previous definition that 0 would be divisible by 0 however there is no result of the operation $0 : 0!$ That is why the number m must be positive.

For example 12 is divisible by 4 and by 6; ($12 = 4 \cdot 3$, $12 = 6 \cdot 2$), but 12 is not divisible by 5; (there exists no positive integer k such that $12 = 5 \cdot k$).

Definition 2. *For a nonnegative integer n , every product $k \cdot n$, where k is a nonnegative integer, is called the multiple of the number n .*

The set of all multiples of n is denoted as W_n . For example:

$$W_0 = \{0\}$$

$$W_1 = \{0, 1, 2, 3, \dots\} = \mathbb{N}$$

$$W_3 = \{0, 3, 6, 9, \dots\} = 3\mathbb{N} - \text{numbers divisible by 3}$$

$$W_{11} = \{0, 11, 22, 33, \dots\} = 11\mathbb{N} - \text{numbers divisible by 11}$$

It is very important (and often forgotten in schoolbooks) that 0 is the least multiple of any positive integer n .

Remark 1. *Since we have restricted our considerations to positive integers, thus the considered notions have close connections.*

If a positive integer m is a divisor of the positive integer n , then n is a multiple of m and conversely, if a positive integer n is a multiple of a positive integer m , then m is a divisor of n .

Divisors of any positive integer can be characterized by the idea of congruence modulo m .

Theorem 2. *If n is a nonnegative integer, then every integer m , which satisfies condition $n = 0$ modulo m , is a divisor of the number n .*

The set of all divisors of the number n is denoted as D_n . For example:

$$D_0 = \{1, 2, 3, \dots\} = \mathbb{N}_+,$$

$$D_1 = \{1\},$$

$$D_3 = \{1, 3\},$$

$$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}.$$

Of course, 1 is a divisor of every positive integer and n is the divisor of n , so every positive integer greater than 1 has at least two divisors.

2. PRIME AND COMPOSED NUMBERS

Definition 3. *A positive integer is called prime number if it is greater than 1 and it is divisible by 1 and itself only (it has only two divisors).*

Definition 4. *A positive integer n is called composed if it is not prime.*

Let us notice that the above definitions have sense only for positive integers greater than 1. Moreover, composed number has a divisor greater than 1 and smaller than n (it has at least three divisors). Hence a positive integer n is composed if it has divisor greater than 1 and not greater than $\lfloor \sqrt{n} \rfloor$. Then we have to search divisors only to $\lfloor \sqrt{n} \rfloor$.

It is not difficult to prove that:

Theorem 3. *The set of prime numbers is infinite.*

The proof of this theorem is quite simple: If this set would be finite, then it would be represented as a finite sequence (a_1, \dots, a_n) . Then the number $a_1 \cdot \dots \cdot a_n + 1$ would be prime, since it would not be divisible by any of all previous prime numbers. A contradiction proves the theorem.

Theorem 4. *Every composed positive integer can be represented in only one way (if we ignore the order) as a product of prime numbers.*

This theorem is connected with the problem of representing any positive integer as a product of prime numbers. But before we are able to represent a number in such a form, it will be useful to find all divisors of a number. It will be also needed to find all prime numbers less than a given number. It is possible to write small computer program, which can find prime numbers (less than a given number). We present such programs written using Python language.

3. SIEVE OF ERATOSTHENES

The method of Eratosthenes is one of the best and fastest methods of finding prime numbers. It is also important because of it is easy writing computer programs finding prime numbers smaller than n . This algorithm has complexity $O(n \log n)$ so it is linear-logarithmic but algorithms finding prime numbers via checking all divisors of positive integers less than n usually has complexity $O(n\sqrt{n})$. Let us show both of those algorithms and examples of programs writing in PYTHON language. Here you can find schemes and programs for finding prime numbers less than a given number n . In all programs and schemes operation $a\%b$ means remainder from the division a by b . For example, $6\%4 = 2$.

FINDING PRIME NUMBERS - SLOW PROGRAM IN PYTHON LANGUAGE

```
import math
print("Determine n")
n=int(input())
for i in range (2,n):
    d=2
    while i%d!=0 and d<=math.floor(math.sqrt(i)):
        d=d+1
    if d==math.floor(math.sqrt(i))+1:
        print(i)
input()
```

SIEVE OF ERATOSTHENES - PROGRAM IN PYTHON LANGUAGE

```
print("Determine n")
n=int(input())
a=[0 for i in range (n)]
for i in range (2,n):
    if a[i]==0:
        print(i)
        k=2
        while (i*k)<n:
            a[i*k]=1
            k=k+1
input()
```

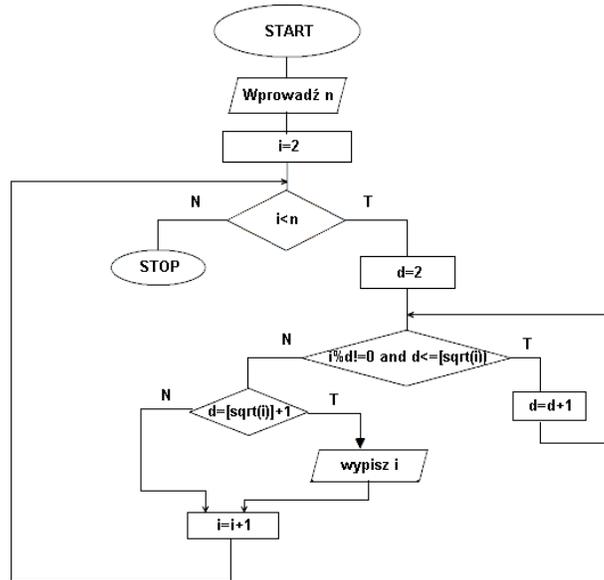


FIGURE 1. Finding prime numbers - slow algorithm

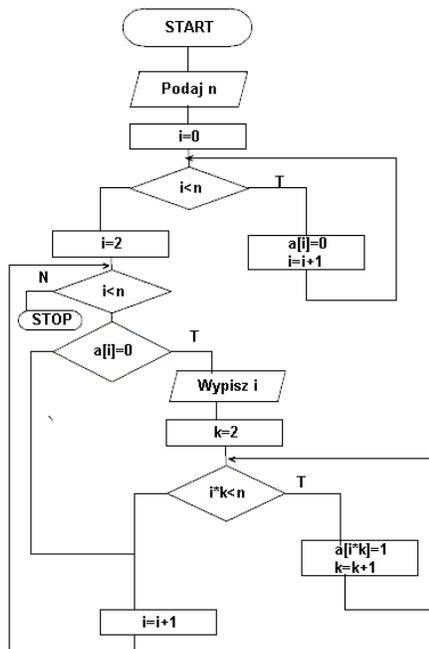


FIGURE 2. Finding prime numbers - Sieve of Eratosthenes

4. LEAST COMMON MULTIPLE AND GREATEST COMMON DIVISOR

Talking on prime and composed numbers it is worth saying about least common multiple and greatest common divisor of two given numbers. We noticed that 0 is the least natural number which is common multiple of any natural number, but it is not a good message for the purpose of common multiples. That is why we consider only positive multiples of positive integers.

Definition 5. *The least common multiple (shortly LCM) of two positive integers is the smallest positive integer that is divisible by both of them.*

Hence

$$W_{12} = \{0, 12, 24, 36, 48, 60, 72, \dots\}$$

$$W_{16} = \{0, 16, 32, 48, 64, 80, 96, \dots\}$$

Then $LCM(12, 16) = 48$.

In school practice LCM is usually computed by factorization onto prime numbers. For example:

	12	16
2	6	8
2	3	4
2	–	2
2	–	1
3	1	–

Then $LCM(12, 16) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 48$.

The method which was presented is not the best one. Much better method is based on the following theorem:

Theorem 5. *Let n and m be two positive integers such that $n \leq m$. If k_0 is the least element of the set $\{k \in \mathbb{N} : n|(km)\}$, then $LCM(n, m) = k_0 \cdot m$.*

Example 1. Let us consider numbers 24 and 9. We have $24 \cdot 1 = 24$ but 24 is not divisible by 9, $24 \cdot 2 = 48$ but 48 is not divisible by 9. Finally $24 \cdot 3 = 72$ and 72 is divisible by 9 so $LCM(24, 9) = 72$.

Definition 6. *The greater common divisor (GCD) of two positive integers is the greatest number which is the divisor of both of them.*

For example:

$$D_{12} = \{1, 2, 3, 4, 6, 12\},$$

$$D_{16} = \{1, 2, 4, 8, 16\}.$$

Then $GCD(12, 16) = 4$.

We can observe that this time greatest common divisor of two positive integers is a positive number. It is worth saying that any nonnegative integer is a divisor of 0, thus $GCD(0, n) = n$ for any nonnegative integer n .

There is another theorem which allows to compute LCM with the aid of GCD .

Theorem 6. *If n and m are two positive integers then*

$$LCM(n, m) \cdot GCD(n, m) = n \cdot m.$$

The above equality is equivalent to any one of the following ones:

$$LCM(n, m) = \frac{nm}{GCD(n, m)}$$

$$GCD(n, m) = \frac{nm}{LCM(n, m)}$$

In school practice GCD is usually calculated by factorization onto prime numbers. For example:

	12	16
2	6	8
2	3	4
2	–	2
2	–	1
3	1	–

Then $GCD(12, 16) = 2 \cdot 2 = 4$.

This is not the best way of finding GCD of two numbers, especially if those numbers are pretty large. Much better manner is described by Euclid's algorithm. We remind it in the following.

Let m and n be two positive integers such that $n < m$. Let us denote them in the following way:

$$m = n_0, \quad n = n_1.$$

From theorem of dividing positive integers with the remainder, there are two nonnegative integers k_1 and n_2 such that

$$n_0 = k_1 \cdot n_1 + n_2, \quad \text{where } n_2 \in \{0, 1, \dots, n_1 - 1\}.$$

Following that way we can state that there exists a sequence (n_0, n_1, \dots) such that

$$n_{p+1} = k_p \cdot n_p + n_{p+1}, \quad \text{where } n_{p+1} \in \{0, 1, \dots, n_p - 1\}.$$

The obtained sequence (n_p) consists of nonnegative integers and is decreasing, hence it must be finite. The last positive member of this sequence is equal to $GCD(m, n)$.

Here you can find a scheme and program for calculating *GCD* with application of Euclid's algorithm.

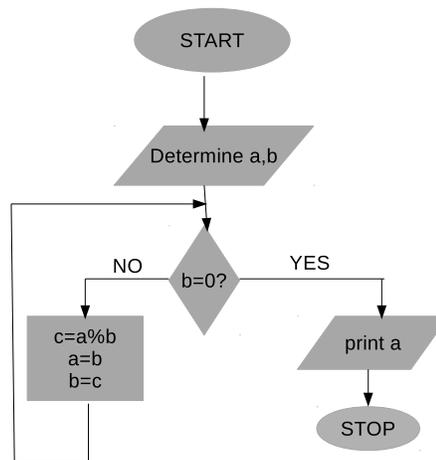


FIGURE 3. **GCD - Euclid's algorithm**

EUCLID'S ALGORITHM - PROGRAM IN PYTHON LANGUAGE

```

print("Determine two positive integers")
a,b=int(input()),int(input())
while b!=0:
    c=a%b
    a=b
    b=c
print("GCD of the above numbers equals",a)
input()
  
```

5. PERFECT NUMBERS, AMICABLE NUMBERS

For pupils which are interested in mathematics, we can enlarge some topics on number theory connected with divisibility. For example, perfect numbers and amicable numbers are very interesting.

Definition 7. *A positive integer n is called a perfect number if it is equal to the sum of all its divisors less than n .*

Perfect numbers state the big mystery in number theory because of:

- We don't know whether the set of perfect numbers is finite or infinite.
- All found perfect numbers, which are known, are even; we don't know if there is any odd one.
- There is no effective algorithm of finding perfect numbers. But there is known a connection between perfect and Mersenne's prime numbers.

First 7 perfect numbers are as follows:

6, 28, 496, 8128, 33550336, 8589869056, 137438691328.

Definition 8. *Two different positive integers are amicable if each of them is the sum of divisors of the other one (less than this one).*

There is a similar problem with amicable numbers. We don't know if there is finite or infinite set of pairs of them and all found amicable pairs are both odd or both even; we don't even know if there is a pair of amicable numbers such that one number is odd and the other one is even.

A few of pairs of amicable numbers are as follows:

(220, 284), (1184, 1210), (2620, 2924), (5020, 5564), (6232, 6368),
(10744, 10856), (12285, 14595).

6. A FEW EXERCISES AND PROBLEMS FOR HOMEWORK

Problem 1. *Calculate (applying Euclide's algorithm) $GCD(240, 600)$ and then $LCM(600, 240)$.*

Problem 2. *Find all divisors of number 1200. Apply the system of writing them in pairs.*

Problem 3. *Find all multiples of 13 which are in the interval $[600, 800]$.*

Solution: $\lfloor \frac{600}{13} \rfloor = 46$, $\lfloor \frac{800}{13} \rfloor = 61$. So we have:

$W_{13[600,800]} = \{47 \cdot 13, 48 \cdot 13, \dots, 61 \cdot 13\} = \{611, 624, 637, \dots, 793\}$.

Problem 4. *Is it true that if a positive integer is divisible by 4 and by 6 then it is divisible by 24?*

One can prove the following theorem:

Theorem 7. *If a positive integer a is divisible by b and it is also divisible by c and additionally $GCD(b, c) = 1$ then a is divisible by $b \cdot c$.*

We can apply this theorem to creating new characteristics about divisibility by composed numbers.

Problem 5. (1) *How to find principle divisibility of 6?*

(2) *How to find principle divisibility of 24?*

(3) *How many two-digit numbers are divisible by 7?*

(4) *How many three-digit numbers are divisible by 7?*

$$L = \lfloor \frac{999}{7} \rfloor - \lfloor \frac{99}{7} \rfloor = 142 - 14 = 128.$$

(5) *How many two-digit numbers are divisible by 4 and by 6?*

$$\text{Since } LCM(4, 6) = 12, \text{ then } L = \lfloor \frac{99}{12} \rfloor - \lfloor \frac{9}{12} \rfloor = 8$$

(6) *How many two-digit numbers are divisible by 4 or by 6?*

$$L = \lfloor \frac{99}{4} \rfloor - \lfloor \frac{9}{4} \rfloor + \lfloor \frac{99}{6} \rfloor - \lfloor \frac{9}{6} \rfloor - (\lfloor \frac{99}{12} \rfloor - \lfloor \frac{9}{12} \rfloor) = 24 - 2 + 16 - 1 - 8 = 29.$$

How to find the number of divisors?

Problem 6. *Calculate, how many divisors the numbers 240, 125, and 3000000 have.*

Solution: Of course every prime number has only two divisors. If we take into account composed number n we first have to present it as the product of prime numbers:

$$n = \prod_{i=1}^n p_i^{k_i}.$$

It can be easily proved that the number of divisors is equal to:

$$L = \prod_{i=1}^n (k_i + 1). \tag{1}$$

For example, we want to know how many divisors the number 6000 has. Factorizing the number 6000:

$$6000 = 2^4 \cdot 3^1 \cdot 5^3.$$

Then, on the base of (1) we obtain:

$$L = (4 + 1)(1 + 1)(3 + 1) = 40.$$

Now we can present some simple tasks.

(1) A positive integer divided by 5 gives a remainder 4. What remainder shall we get if we divide square of this number by 5?

- (2) First of 2 numbers divided by 6 gives a remainder 4 and the other one divided also by 6 gives a remainder 3. What remainder shall we get if we divide the sum of these numbers by 6 and what remainder shall we get if we divide product of these numbers by 6?
- (3) Show some practical tasks in which we use GCD or LCM.
- (4) Prove that the sum of two odd numbers is even.
- (5) Prove that the product of three consecutive natural numbers is divisible by 6.
- (6) Find all pairs of positive integers which are solutions of the following equations:
 - (a) $2x + y = 8$
 - (b) $3x + 5y = 8$
- (7) Find all positive integers n for which the fraction $\frac{2n+16}{n+2}$ is a positive integer.

REFERENCES

- [1] J. Jędrzejewski, A. Vizvary, M. Ziółkowski, *Matematyka - świat liczb - poradnik dla nauczyciela gimnazjum klasa I*. Res Polonia. Łódź 2012.
- [2] J. Jędrzejewski, A. Vizvary, M. Ziółkowski, *Matematyka - świat liczb - podręcznik dla uczniów gimnazjum klasa I*. Res Polonia. Łódź 2012.
- [3] J. Jędrzejewski, A. Vizvary, M. Ziółkowski, *Matematyka - świat liczb - zbiór zadań dla uczniów gimnazjum klasa I*. Res Polonia. Łódź 2012.
- [4] J. Jędrzejewski, A. Vizvary, M. Ziółkowski, *Repetitorium z matematyki*. Res Polonia. Łódź 2012.

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