

$M/M/n/m$ queueing system with non-identical servers

Marcin Ziółkowski

*Institute of Mathematics and Computer Science
 Jan Długość University of Częstochowa
 al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland
 e-mail: m.ziolkowski@ajd.czyst.pl*

Abstract. $M/M/n/m$ queueing systems with identical servers are well known in queueing theory and its applications. The analysis of these systems is very simple thanks to the fact that the number of customers $\eta(t)$ in the system at arbitrary time instant t forms a Markov chain. The main purpose of the paper is to analyse $M/M/n/m$ system under assumption that its servers are different, i.e. they have different parameters of service time.

1. The analysis of classical $M/M/n/m$ queueing system

Let us consider the classical $M/M/n/m$ queueing system ($m < \infty$). Let a be an arrival rate of customers, μ – the parameter of service time distribution. This system is illustrated on fig. 1.

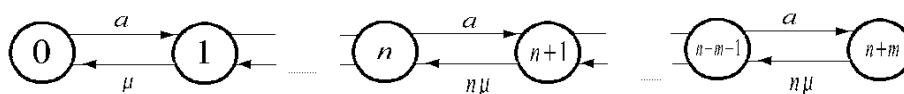


Figure 1: Graph of the classical $M/M/n/m$ queueing system ($m < \infty$)

Denote $\rho = a/(n\mu)$. In the case of $\rho < \infty$ the steady state exists and we have the following equations for steady state probabilities $p_k = P\{\eta = k\}$:

$$\begin{cases} ap_{k-1} = k\mu p_k, & k = \overline{1, n}, \\ ap_{k-1} = n\mu p_k, & k = \overline{n+1, n+m}, \end{cases} \quad (1)$$

where η is the stationary number of customers present in the system, $k = 0, 1, \dots$

It can be easily shown ([1]) that the solution of this system has the form:

$$p_k = \begin{cases} \frac{(n\rho)^k}{k!} p_0, & k = \overline{1, n}, \\ \frac{n^n \rho^k}{n!} p_0, & k = \overline{n+1, n+m}. \end{cases} \quad (2)$$

The value $p_0 = P\{\eta = 0\}$ is obtained from the normalization condition $\sum_{k=0}^{n+m} p_k = 1$. If $m = \infty$ the graph of the system is similar. In this case $\eta(t)$ is a birth-death process and we can also obtain p_k probabilities (if $\rho < 1$) from analogous equations. The final formulas are also very similar.

Assume now that μ_i is a service time distribution parameter of i -th server in the system, $i = \overline{1, n}$ and these parameters may be different. The problem appeared for the first time in [2],[3] and [4]. In this case it is possible that arriving customer has to choose one of free different servers. Hence we must define the discipline of such choice. We shall analyse the following disciplines:

1. RANDOM CHOICE.

If there are k free servers, the arriving customer chooses every free server with the same probability $\frac{1}{k}$. Such systems will be denoted as $M/M/n/m - NHRC$ (non-homogeneous servers with classical random choice)

2. FASTEST SERVER CHOICE.

Arriving customer chooses the fastest server. Such systems will be denoted as $M/M/n/m - NHFC$ (non-homogeneous servers with fastest server choice).

We shall describe the behaviour of the system by the following generalized markovian process:

$$(\eta(t), i_1(t), i_2(t), \dots, i_l(t)), \quad (3)$$

where $l = \min(\eta(t), n)$ and $i_1(t), i_2(t), \dots, i_l(t)$ is the sequence of the numbers of busy servers written increasingly. If $\eta(t) = 0$, the process reduces to $\eta(t)$.

Process (3) is characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \quad (4)$$

$$P_{k f_1 f_2 \dots f_l}(t) = P\{\eta(t) = k, i_1(t) = f_1, i_2(t) = f_2, \dots, i_l(t) = f_l\}, \quad (5)$$

$$k = \overline{1, n+m}.$$

It is clear that if $k \geq n$ then process $\eta(t)$ can be understood as a birth-death process and we can rewrite functions from (5) as $P_k(t) = P\{\eta(t) = k\}$. If $k < n$ we have

$$P_k(t) = P\{\eta(t) = k\} = \sum_{\{F_k^n\}} P_{kf_1f_2\dots f_k}(t), \tag{6}$$

where $\{F_k^n\}$ is the set of all combinations of the set $\{f_1, f_2, \dots, f_k\}$.

Now we shall investigate these systems in some special cases and try to find formulas for p_k in general case.

2. Analysis of $M/M/2/m - NHRC$ and $M/M/3/m - NHRC$ queueing systems

Let us consider the generalization of $M/M/n/m$ queueing system with two non-identical servers and random choice of a free server. Denote as μ_1, μ_2 service time parameters for the servers. We will also use the notation: $\rho = \frac{a}{\mu_1 + \mu_2}$. Graph of such system is presented on fig. 2.

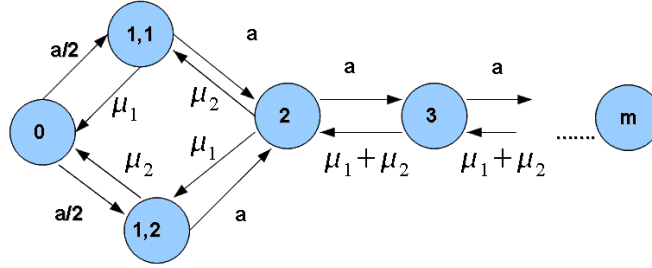


Figure 2: Graph of the $M/M/2/m - NHRC$ queueing system ($m < \infty$)

To describe behaviour of the system we introduce the following markovian process:

$$(\eta(t), i_1(t), \dots, i_l(t)), \tag{7}$$

where $l = \min(\eta(t), 2)$.

The process (7) is characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \tag{8}$$

$$P_{11}(t) = P\{\eta(t) = 1, i_1(t) = 1\}; \tag{9}$$

$$P_{12}(t) = P\{\eta(t) = 1, i_1(t) = 2\}; \tag{10}$$

$$P_k(t) = P\{\eta(t) = k\}, \quad k = \overline{2, m+2}. \tag{11}$$

The Kolmogorov equations for these functions have the form:

$$P'_0(t) = -aP_0(t) + \mu_1 P_{11}(t) + \mu_2 P_{12}(t); \quad (12)$$

$$P'_{11}(t) = -(a + \mu_1)P_{11}(t) + \frac{a}{2}P_0(t) + \mu_2 P_2(t); \quad (13)$$

$$P'_{12}(t) = -(a + \mu_2)P_{12}(t) + \frac{a}{2}P_0(t) + \mu_1 P_2(t); \quad (14)$$

$$P'_2(t) = -(a + \mu_1 + \mu_2)P_2(t) + a(P_{11}(t) + P_{12}(t)) + (\mu_1 + \mu_2)P_3(t); \quad (15)$$

$$P'_k(t) = -(a + \mu_1 + \mu_2)P_k(t) + aP_{k-1}(t) + (\mu_1 + \mu_2)P_{k+1}(t), \quad k = \overline{3, m+1}; \quad (16)$$

$$P'_{m+2}(t) = -(\mu_1 + \mu_2)P_{m+2}(t) + aP_{m+1}(t); \quad (17)$$

$$P_0(t) + P_{11}(t) + P_{12}(t) + \sum_{k=2}^{m+2} P_k(t) = 1. \quad (18)$$

In steady state (if $\rho < \infty$) we obtain the following equations for probabilities p_0, p_{11}, p_{12} and p_k that are limits of functions (8)–(11) if $t \rightarrow \infty$:

$$ap_0 = \mu_1 p_{11} + \mu_2 p_{12}; \quad (19)$$

$$(a + \mu_1)p_{11} = \frac{a}{2}p_0 + \mu_2 p_2; \quad (20)$$

$$(a + \mu_2)p_{12} = \frac{a}{2}p_0 + \mu_1 p_2; \quad (21)$$

$$(a + \mu_1 + \mu_2)p_2 = a(p_{11} + p_{12}) + (\mu_1 + \mu_2)p_3; \quad (22)$$

$$(a + \mu_1 + \mu_2)p_k = ap_{k-1} + (\mu_1 + \mu_2)p_{k+1}, \quad k = \overline{3, m+1}; \quad (23)$$

$$(\mu_1 + \mu_2)p_{m+2} = ap_{m+1}; \quad (24)$$

$$p_0 + p_{11} + p_{12} + \sum_{k=2}^{m+2} p_k = 1. \quad (25)$$

The solution has the form:

$$p_{11} = \frac{a}{2\mu_1}p_0; \quad (26)$$

$$p_{12} = \frac{a}{2\mu_2}p_0; \quad (27)$$

$$p_1 = p_{11} + p_{12} = \frac{ap_0}{2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right); \quad (28)$$

$$p_k = \frac{ap_0}{2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \rho^{k-1}, \quad k = \overline{2, m+2}. \quad (29)$$

Formulas (28)–(29) can be rewritten as

$$p_k = \frac{ap_0}{2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \rho^{k-1}, \quad k = \overline{1, m+2}, \quad (30)$$

where value of p_0 can be obtained from the normalization condition (25):

$$p_0 = \left[1 + \frac{a}{2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \sum_{k=1}^{m+2} \rho^{k-1} \right]^{-1}. \quad (31)$$

If we investigate $M/M/2/\infty - NHRC$ system in steady state ($\rho < 1$), formulas for p_k probabilities (30) will be the same, but the relation (31) will have simpler form (in this case we use formula for the sum of geometric series).

If we analyse $M/M/3/m - NHRC$ system ($\rho = \frac{a}{\mu_1 + \mu_2 + \mu_3}$) and write analogous equations we will obtain the following results:

$$p_1 = \frac{ap_0}{3} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \right); \quad (32)$$

$$p_k = \frac{a^2 p_0}{6} \left(\frac{1}{\mu_1 \mu_2} + \frac{1}{\mu_1 \mu_3} + \frac{1}{\mu_2 \mu_3} \right) \rho^{k-2}, \quad k = \overline{2, m+3}; \quad (33)$$

$$p_0 = \left[1 + \frac{a}{3} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \right) + \frac{a^2}{6} \left(\frac{1}{\mu_1 \mu_2} + \frac{1}{\mu_1 \mu_3} + \frac{1}{\mu_2 \mu_3} \right) \sum_{k=2}^{m+3} \rho^{k-2} \right]^{-1}. \quad (34)$$

Analysis of $M/M/n/m - NHRC$ for $n > 3$ is more complicated because the number of equations is increasing exponentially and it is difficult to find exact solution. But obtained results let predict general solution for the arbitrary number of servers in the following form:

$$p_k = \begin{cases} \frac{a^k (n-k)! p_0}{n!} \sum_{\{F_k^n\}} \frac{1}{\prod_{x_i \in F_k^n} \mu_{x_i}}, & k = \overline{1, n-1}, \\ \frac{a^k (n-k)! p_0}{n!} \sum_{\{F_n^n\}} \frac{1}{\prod_{x_i \in F_n^n} \mu_{x_i}} \rho^{k-n}, & k = \overline{n, n+m}, \end{cases} \quad (35)$$

where F_k^n denotes k -element subset of n element set.

3. Analysis of $M/M/2/m - NHFC$ queueing system

In this case, if we assume that $\mu_1 > \mu_2$, then we will not have transition $0 \rightarrow (1, 2)$ in the graph presented on fig. 2. and transition $0 \rightarrow (1, 1)$ parameter will be equal a .

So in this case we have the following equations in the steady state:

$$ap_0 = \mu_1 p_{11} + \mu_2 p_{12}; \quad (36)$$

$$(a + \mu_1)p_{11} = ap_0 + \mu_2 p_2; \quad (37)$$

$$(a + \mu_2)p_{12} = \mu_1 p_2. \quad (38)$$

The rest of equations for functions p_k will be the same as in (22)–(25).

The solution of this system has the form:

$$p_k = \frac{a(a + \mu_2)(\mu_1 + \mu_2)p_0}{\mu_1\mu_2(2a + \mu_1 + \mu_2)}\rho^{k-1}, \quad k = \overline{1, m+2}; \quad (39)$$

$$p_0 = \left[1 + \frac{a(a + \mu_2)(\mu_1 + \mu_2)}{\mu_1\mu_2(2a + \mu_1 + \mu_2)} \sum_{k=1}^{m+2} \rho^{k-1} \right]^{-1}. \quad (40)$$

General analysis for $n \geq 3$ is complicated and predicting of general formulas for p_k is not easy.

References

- [1] O. Tikhonenko. *Metody probabilistyczne analizy systemów informacyjnych*. Akademicka Oficyna Wydawnicza EXIT, Warszawa, 2006.
- [2] H. Gumbel. *Waiting lines with heterogeneous servers*. Operations Research, vol. 8, no. 4, pp. 504-511, 1960.
- [3] V.P. Singh. *Markovian queues with three heterogeneous servers*. AIIE Transactions, vol. 3, no. 1, pp. 45-48, 1971.
- [4] V.P. Singh. *Two-server Markovian queues with balking: Heterogeneous vs. homogeneous servers*. Operation Research, vol. 18, no. 1, pp. 145-159, 1970.